International Journal of Theoretical Physics, Vol. 27, No. 3, 1988

Stability of the General Exponent of Nonlinear Impulsive Differential Equations in a Banach Space

D. D. Bainov,[1] P. P. Zabreiko, and S. T. Kostadinov

Received August 14, 1987

A theorem is proved which gurantees stability under small perturbations of the general exponent of impulsive nonlinear systems in a Banach space.

1. INTRODUCTION

Recently, in relation to numerous applications, the theory of impulsive differential equations has begun to develop rapidly. The first work dedicated to this subject was Millman and Myshkis (1960). Other work in this field includes Myshkis and Samoilenko (1967), Samoilenko and Perestink (1977), and Simeonov and Bainov (1985).

In the present paper the general exponent of impulsive equations of a special type is considered and conditions for its stability under quite natural assumptions are found.

2. STATEMENT OF THE PROBLEM

Let X be a complex Banach space. Consider the impulsive equation

$$dx/dt = f(t, x) \qquad (t \neq t_n) \tag{1}$$

$$x(t_n+0) = Q_n x(t_n - 0) \qquad (n = 1, 2, \ldots) \tag{2}$$

where $t_1 < t_2 < \cdots$ are fixed times of impulses satisfying the condition $\lim_{n\to\infty} t_n = \infty$.

[1]P.O. Box 45, 1504 Sofia, Bulgaria.

0020-7748/88/0300-0373$06.00/0

We shall say that conditions (H) are fulfilled if the following conditions hold:

H1. The function $f:[0,\infty)\times X\to X$ is continuous and $f(t,0)=0$.

H2. For $t_0\geq 0$ the Cauchy problem

$$dx/dt=f(t,x),\qquad x(t_0)=x_0 \tag{3}$$

has a unique solution $x(t)$, which is defined for $t\geq t_0$.

H3. The operators Q_n, $n=1,2,\ldots$, are linear, bounded, and map X into X.

Definition 1. We shall call a solution of the impulsive equation (1), (2) with initial condition (3) a piecewise continuous function $x(t)$ $(t\geq t_0)$ with discontinuities of first type at the points t_n $(n=1,2,\ldots)$ such that

$$dx/dt=f(t,x(t))\qquad (t\neq t_n)$$

$$x(t_n+0)=Q_n x(t_n-0)\qquad (n=1,2,\ldots)$$

We assume that at the points t_n $(n=1,2,\ldots)$ the function $X(t)$ is left continuous.

Lemma 1. Let conditions (H) be satisfied.

Then for $(t_0,x_0)\in[0,\infty)\times X$ the Cauchy problem for the impulsive equation (1), (2) with initial condition (3) has a unique solution which is defined on $[t_0,\infty)$ and satisfies the integral equation

$$x(t)=Q(t,t_0)x_0+\int_{t_0}^{t}Q(t,s)f(s,x(s))\,ds \tag{4}$$

where

$$Q(t,\tau)=\prod_{\tau\leq t_k<t}Q_k \tag{5}$$

The proof of Lemma 1 is carried out on each interval $[t_n, t_n+1)$ by standard methods for nonimpulsive ordinary differential equations.

By means of the equality

$$U(t,t_0)x_0=x(t;t_0,x_0) \tag{6}$$

where $X(t;t_0,x_0)$ is a solution of the Cauchy problem for the impulsive equation (1), (2) with initial condition (3), we can define a two-parameter family of operators $U(t,t_0)$ $(0\leq t_0\leq t<\infty)$ mapping X into itself. This two-parameter family of operators possesses the semigroup property

$$U(t,\tau)U(\tau,t_0)=U(t,t_0)(0\leq t_0\leq\tau\leq t<\infty) \tag{7}$$

but it is not a continuous operator-valued function, since

$$U(t_n+0,\tau)=Q_nU(t_n-0,\tau)\qquad (t_n>\tau) \tag{8}$$

and, analogously,

$$U(t, t_n+0) = Q_n U(t, t_n - 0) \qquad (t > t_n) \tag{9}$$

Definition 2. We call a general exponent $\varkappa(r)$ of the impulsive equation (1), (2) the greatest lower bound of the number δ for which there exist numbers τ and N such that for any solution $x(t)$ satisfying

$$\|x(t_0)\| \le r \tag{10}$$

the following inequality holds as well:

$$\|x(t)\| \le N e^{\delta(t-\tau)} \|x(\tau)\| \qquad (t_0 \le \tau \le t < \infty) \tag{11}$$

If such a number δ does not exist, we set $\varkappa(r) = \infty$.

We note that in the case considered the classical equality

$$\varkappa(r) = \sup_{\|x(t_0)\| \le r} \ \overline{\lim_{\tau, t-\tau \to \infty}} \ \frac{\ln \|x(t)\| / \|x(\tau)\|}{t - \tau}$$

is not always valid, since the solutions of the impulsive equations (1), (2) are piecewise continuous; therefore, they are not continuously differentiable.

The numbers N in (11) have to satisfy the inequalities

$$\|Q_n\| \le N \qquad (n = 1, 2, \ldots) \tag{12}$$

Let us consider the conditions under which the impulsive equations (1), (2) has a finite general exponent.

Lemma 2. Let the following conditions be fulfilled:

1. $\|Q(t, \tau)\| \le M e^{\delta(t-\tau)} \qquad (0 \le \tau \le t \le \infty)$

where M and δ are constants.

2. $\|f(t, x)\| \le \mu(t) \|x\|$

where the function $\mu(t)$ $(t \ge 0)$ is such that

$$\exp\left[M \int_\tau^t \mu(s)\, ds \right] \le N \exp[\gamma(t - \tau)] \qquad (0 \le \tau \le t < \infty) \tag{13}$$

and N and γ are constants.

3. Condition H2 holds.

Then the general exponent $\varkappa(r)$ satisfies the inequality

$$\kappa(r) \le \delta + \gamma \tag{14}$$

Proof. By Lemma 1, from (4), for $0 \le \tau \le t < \infty$ we have

$$x(t) = Q(t, \tau) x(\tau) + \int_\tau^t Q(t, s) f(s, x(s))\, ds$$

From conditions 1 and 2 of Lemma 2 we have

$$\|x(t)\| \leq M e^{\delta(t-\tau)}\|x(\tau)\| + M \int_{\tau}^{t} e^{\delta(t-s)}\mu(s)\|x(s)\|\, ds$$

Applying the lemma of Gronwall-Bellman to the last inequality, we obtain

$$\|x(t)\| \leq M\{\exp[\delta(t-\tau)]\}\|x(\tau)\| \exp\left[M \int_{\tau}^{t} \mu(s)\, ds\right]$$

The assertion of Lemma 2 follows from condition (13).

Estimate (14) is rather rough. It does not always allow one to establish when the general exponent is negative. This is possible if, for instance, it is possible to pass from the initial equation (1), (2) to the impulsive equation

$$dz/dt = cz + e^{ct}f(t, e^{-ct}z) \qquad (t \neq t_n) \tag{15}$$

$$z(t_n+0) = Q_n z(t_n) \qquad (n = 1, 2, \ldots) \tag{16}$$

with a positive parameter c, the solutions of which are related to the solutions of (1), (2) by

$$z(t) = e^{ct}x(t)$$

The verification of condition 1 of Lemma 2 is in general a rather complicated problem. In the particular case when the sequence $\{t_n\}$ is an arithmetic progression $t_n = a + nh$ $(n = 1, 2, \ldots; 0 \leq a \leq h)$ and $Q_n = Q$ $(n = 1, 2, \ldots)$, condition 1 of Lemma 2 is satisfied for an arbitrary $\delta > h^{-1} \ln \rho(Q)$, where $\rho(Q)$ is the spectral radius of the operator Q.

Consider the more general case when the sequences $\{t_n\}$ and $\{Q_n\}$ are periodic, i.e., for some constants k and T for $n = 0, 1, \ldots$ the conditions $t_{n+k} = t_n + T$, $Q_{n+k} = Q_n$ hold. In this case condition 1 holds for any $\delta > T^{-1} \ln \rho(Q_1, \ldots, Q_k)^{1/k}$. For the general case simple estimates of δ are unknown to the authors. Condition 2 of Lemma 2 is verified by means of standard methods of mathematical analysis.

Remark 1. If the conditions of Lemma 2 are satisfied, the general exponents of all solutions of the impulsive equation (1), (2) are estimated by one and the same number.

Lemma 3. Let conditions (H) hold. Let us assume, moreover, that there exists a positive number h such that for some $r > 0$ the operator-valued function $U(t, \tau)$ satisfies the conditions

(i) $\|U(t, \tau)x\| \leq M \cdot \|x\| \qquad (|t-\tau| \leq h, \|x\| \leq r)$

(ii) $\|U(t+h, t)x\| \leq C \cdot \|x\| \qquad (0 \leq t < \infty, \|x\| \leq r)$

where $C < 1$ and M are constants.

Then the general exponent $\varkappa(\rho)$ $(\rho = M^{-1}r)$ of the impulsive equation (1), (2) satisfies the condition

$$\kappa(\rho) \leq h^{-1} \ln C \tag{17}$$

Proof. If $\|x_0\| \leq r$ and $t = \tau + nh + \theta$ for some positive integer n and $0 \leq \theta < h$, then by the semigroup property (7) and conditions 1 and 2 of Lemma 3 we obtain the estimate

$$|u(t, \tau)x_0\| \leq MC^n \|x_0\| \leq \frac{M}{C} \Big\{ \exp[h^{-1} \ln C(t-\tau)]\} \|x_0\|$$

which implies (17).

Lemma 3 is proved.

Remark 2. Note that if $\delta > \varkappa(r)$, then for some N the following inequality holds:

$$\| U(t, \tau)x_0\| \leq N e^{\delta(t-\tau)} \|x_0\|$$

If we set $C = M e^{\delta h}$, then we obtain conditions 1 and 2 of Lemma 3; moreover, if we choose h large enough, then the respective estimate for $\varkappa(r)$ will be close enough to δ.

3. MAIN RESULTS

We consider the question of the existence of a finite general exponent of the impulsive equation

$$dx/dt = f(t, x) + g(t, x) \tag{18}$$

$$x(t_n + 0) = Q_n x(t_n) \tag{19}$$

where $g\colon [0, \infty) \times X \to X$.

Definition 3. The function $g(t, x)$ belongs to the class $G(f, \nu, r)$ if, for equation (18), the assertion of the local theorem for the existence of a solution is valid (see, e.g., Daleckii and Krein, 1974) and if, moreover, the following conditions holds:

$$\|g(t, x)\| \leq \nu \|x\| \qquad (\|x\| \leq r) \tag{20}$$

Theorem 1. Let the following conditions be satisfied:

1. Conditions (H) hold.
2. The function $f(t, x)$ satisfies the Lipschitz condition with a constant $\mu > 0$:

$$|f(t, x_1) - f(t, x_2)\| \leq \mu \|x_1 - x_2\| \qquad (\|x_1\|, \|x_2\| \leq r)$$

3. $\|Q(t, \tau)\| \leq M e^{\delta(t-\tau)}$, where $\delta < 0$.

4. There exists a general exponent $\varkappa_0(r)$ of the impulsive equation (1), (2); moreover, $\varkappa_0(r) < 0$.

Then for any $\varepsilon > 0$ there exists numbers ρ $(0 < \rho \leq r)$ and ν depending only on the function $f(t, x)$ and one the operators Q_n $(n = 1, 2, \ldots)$ such that the impulsive equation (18), (19) for $g(t, x) \in G(f, \nu, r)$ has a finite general exponent $\varkappa(\rho)$ satisfying the estimate $\varkappa(\rho) \leq \varkappa_0(r) + \varepsilon$.

Proof. In the proof of Theorem 1 we shall use Lemma 3. Let $\varepsilon > 0$ be so small that $\tilde{\lambda} = \varkappa_0(r) + \varepsilon < 0$ and $\varkappa_0(r) < \lambda < \tilde{\lambda}$. By Definition 2 there exists a number N such that each solution $\xi(t)$ of the impulsive equation (1), (2) satisfying the condition $\|\xi(\tau)\| < r$ also satisfies the inequality

$$\|\xi(t)\| \leq N e^{\lambda(t-\tau)} \|\xi(\tau)\| \qquad (\tau \leq t \leq \infty) \tag{21}$$

Let $h > 0$ be an arbitrary number such that

$$N e^{(\lambda - \tilde{\lambda})h} < 1 \tag{22}$$

We shall show that the impulsive equation (18), (19) satisfies the conditions of Lemma 3 for $C = e^{\lambda h}$, whence the proof of Theorem 1 will follow.

Let $\nu > 0$ be chosen in such a way that the following inequality holds:

$$\frac{M\nu}{\mu + \nu} e^{(\delta - \tilde{\lambda})h} e^{M\mu h} (e^{M(\mu+\nu)h} - 1) \leq 1 - N e^{(\lambda - \tilde{\lambda})h} \tag{23}$$

and let the number ρ be chosen so that

$$\rho N_h \leq r \tag{24}$$

where

$$N_h = N + \frac{M_\nu}{\mu + \nu} e^{M\mu h} (e^{M(\mu+\nu)h} - 1)$$

We shall show that conditions 1 and 2 of Lemma 3 hold.

Let $\|x(\tau)\| \leq \rho$. Note that for all t such that the solution of the impulsive equation (18), (19) lies in the ball $\|x\| \leq r$ the following integral identity holds:

$$x(t) = Q(t, \tau)x(\tau) + \int_\tau^t Q(t, s) f(s, x(s))\, ds + \int_\tau^t Q(t, s) g(s, x(s))\, ds$$

By conditions 2 and 3 of Theorem 1, since $g(t, x) \in G(f, \nu, r)$, we have

$$\|x(t)\| \leq M e^{\delta(t-\tau)} \|x(\tau)\| + \int_\tau^t M e^{\delta(t-s)} (\mu + \nu) \|x(s)\|\, ds$$

Applying to the last inequality the Gronwall–Bellman lemma, we obtain

$$\|x(t)\| \leq M e^{[\delta + M(\mu+\nu)](t-\tau)} \|x(\tau)\| \tag{25}$$

Denote by $\xi(t)$ this solution of the impulsive system (1), (2) which satisfies the initial condition $\xi(t)=x(\tau)$. Then

$$x(t)-\xi(t)=\int_\tau^t Q(t,s)[f(s,x(s))-f(s,\xi(s)]\,ds+\int_\tau^t Q(t,s)g(s,x(s))\,ds$$

from which, as above, we obtain the inequality

$$\|x(t)-\xi(t)\|\le\int_\tau^t M\,e^{\delta(t-s)}\mu\|x(s)-\xi(s)\|\,ds+\int_\tau^t M\,e^{\delta(t-s)}\nu\|x(s)\|\,ds$$

From the above inequality and (25) for $t\in[\tau,\tau+h]$ we deduce the inequality

$$\|x(t)-\xi(t)\|\le\int_\tau^t M\,e^{\delta(t-s)}\mu\|x(s)-\xi(s)\|\,ds+\frac{\mu_\nu}{\mu+\nu}\,e^{\delta(t-\tau)}\times(e^{M(\mu+\nu)h}-1)\|x(\tau)\|$$

Applying again the Gronwall-Bellman lemma, we obtain

$$\|x(t)-\xi(t)\|\le e^{\delta(t-\tau)}\frac{M\nu}{\mu+\nu}(e^{M(\mu+\nu)h}-1)\|x(\tau)\|\,e^{M\mu h} \tag{26}$$

From (21) and (26) it follows that

$$\|x(t)\|\le\|\xi(t)\|+\|x(t)-\xi(t)\|\le\left[N\,e^{\lambda(t-\tau)}+\frac{M\nu}{\mu+\nu}\,e^{\delta(t-\tau)}\,e^{M\mu h}(e^{M(\mu+\nu)h}-1)\right]\|x(\tau)\|$$

i.e.,

$$\|x(t)\|\le\left[N\,e^{(\lambda-\bar\lambda)(t-\tau)}+\frac{M\nu}{\mu+\nu}\,e^{(\delta-\bar\lambda)(t-\tau)}\,e^{\mu Mh}\times(e^{M(\mu+\nu)h}-1)\,e^{\bar\lambda(t-\tau)}\right]\|x(\tau)\| \tag{27}$$

From inequalities (27) and (24) it follows that

$$\|x(t)\|\le N_h\,e^{\bar\lambda(t-\tau)}\|x(\tau)\|\qquad(\tau\le t\le\tau+h)$$

which implies that the solution $x(t)$ for $t\in[\tau,\tau+h]$ lies in the ball $\|x\|\le r$.

Inequalities (27) and (23) imply the validity of the inequality

$$\|x(\tau+h)\|\le e^{\bar\lambda h}\|x(\tau)\|$$

i.e., conditions 1 and 2 of Lemma 3 hold. The proof of Theorem 1 follows from Lemma 3, whence, in particular, we obtain

$$\varkappa(\rho) \leq \tilde{\lambda} \leq \varkappa(r) + \varepsilon$$

Theorem 1 is proved.

Consider the perturbed impulsive equation

$$dx/dt = f(t, x) + g(t, x) \tag{28}$$

$$x(t_n + 0) = (Q_n + \Delta_n)x(t_n) \tag{29}$$

where Δ_n are linear bounded operators mappings X into itself.

Definition 4. The sequence $\{\Delta_n\}$ belongs to the class $D(L, \delta, \{Q_n\})$ if the following condition holds:

$$\|\tilde{Q}(t, s) - Q(t, s)\| \leq L\, e^{\delta(t-s)} \qquad (0 \leq s \leq t < \infty) \tag{30}$$

where the operator $Q(t, s)$ is defined by equality (5) and

$$\tilde{Q}(t, s) = \prod_{s \leq t_n < t} (Q_n + \Delta_n) \tag{31}$$

Theorem 2. Let the conditions of Theorem 1 hold.

Then for any $\varepsilon > 0$ there exist number ρ, ν, and L depending only on the function $f(t, x)$ and on the operators Q_n so that the impulsive equation (28), (29) for $g(t, x) \in G(f, \nu, r)$ and $\{\Delta_n\} \in D(L, \delta, \{Q_n\})$ has a finite general exponent $\varkappa(\rho)$ such that $\varkappa(\rho) \leq \varkappa_0(r) + \varepsilon$.

The proof of Theorem 2 is a modification of the proof of Theorem 1.

Remark 3. In Theorems 1 and 2 the conditions $\delta < 0$ and $\varkappa_0(r) < 0$ are not essential. By the exponential substitution $x = e^{\alpha t}z$ for α suitably chosen we can reduce the general case to the one when $\varkappa_0(r)$ and δ are negative.

REFERENCES

Daleckii, Ju. L., and Krein, M. G. (1974). *Stability of Solutions of Differential Equations in Banach Space,* AMS, Books and Journals in Advanced Mathematics.

Millman, V. D., and Myshkis, A. D. (1960). On the stability of motion in the presence of impulses, *Sibirski Math. Zhurnal,* **1**(2), 233-237 (in Russian).

Myshkis, A. D., and Samoilenko, A. M. (1967). Systems with impulses in prescribed moments of time, *Math. Sbornik,* **74**(2) (in Russian).

Samoilenko, A. M., and Perestiuk, N. A. (1977). Stability of the solutions of impulsive differential equations, *Differential Equations,* **11**, 1981-1992 (in Russian).

Simeonov, P. S., and Bainov, D. D. (1985). Stability under persistent disturbances for systems with impulse effect, *Journal of Mathematical Analysis and Applications,* **109**(2), 546-563.